MATH 245 S20, Exam 2 Solutions

1. Carefully define the following terms: Proof by Cases Theorem, Nonconstructive Existence Theorem, Proof by Reindexed Induction

   The Proof by Cases theorem says: Let p, q be propositions. Suppose there are propositions $c_1, c_2, \ldots, c_k$ with $c_1 \vee c_2 \vee \cdots \vee c_k \equiv T$. Now, if $(p \wedge c_1) \rightarrow q$, $(p \wedge c_2) \rightarrow q$, $\ldots$, $(p \wedge c_k) \rightarrow q$ are all true, then $p \rightarrow q$ is true. The Nonconstructive Existence Theorem says: If $\forall x \in D,\ \neg P(x) \equiv F$, then $\exists x \in D,\ P(x)$ is true. To prove $\forall x \in \mathbb{N},\ P(x)$ by reindexed induction, we must (a) prove $P(1)$ is true; and (b) prove $\forall x \in \mathbb{N}$ with $x \geq 2$, $P(x-1) \rightarrow P(x)$.

2. Carefully define the following terms: Proof by Minimum Element Induction Thm, well-ordered, big O

   The Proof by Minimum Element Induction Theorem says: Let S be a nonempty set of integers. If S has a lower bound, then it has a minimum. Let S be a set of numbers with some ordering $<$. We say that S is well-ordered by $<$ if every nonempty subset of S has a minimum according to $<$. Let a_n and b_n be sequences. We say that $a_n = O(b_n)$ if $\exists n_0 \in \mathbb{N},\ \exists M \in \mathbb{R},\ \forall n \geq n_0,\ |a_n| \leq M|b_n|$.

3. Let $x \in \mathbb{R}$. Use cases to prove that $|x-2| + |x-5| \geq 3$.

   Case 1: If $x < 2$, then $|x-2|+|x-5| = 2-x+5-x = 7-2x$. We multiply $x < 2$ by -2 to get $-2x > -4$, and add 7 to get $7-2x > 7-4 = 3$. Hence $|x-2|+|x-5| = 7-2x > 3$.

   Case 2: If $2 \leq x \leq 5$, then $|x-2| + |x-5| = x-2+5-x = 3 \geq 3$.

   Case 3: If $x > 5$, then $|x-2|+|x-5| = x-2+x-5 = 2x-7$. We multiply $x > 5$ by 2 to get $2x > 10$, and add -7 to get $2x-7 > 10-7 = 3$. Hence, $|x-2|+|x-5| = 2x-7 > 3$.

   In all three cases, the desired inequality holds.

4. Prove that $\forall x \in \mathbb{R},\ \lfloor -x \rfloor = -\lceil x \rceil$.

   Let $x \in \mathbb{R}$ be arbitrary. Applying the definitions of ceiling and floor, we get $\lfloor -x \rfloor \leq -x < \lfloor -x \rfloor + 1$ and $\lceil x \rceil - 1 < x \leq \lceil x \rceil$. We multiply the latter by -1 to get $-\lceil x \rceil + 1 > -x \geq -\lceil x \rceil$. We now have a choice for how to continue.

   SOLUTION 1: We combine inequalities to get $\lfloor -x \rfloor \leq -x < -\lceil x \rceil + 1$ and $-\lceil x \rceil \leq -x < \lfloor -x \rfloor + 1$. Hence $-\lceil x \rceil - 1 < \lfloor -x \rfloor < -\lceil x \rceil + 1$. Since these are integers, by Thm. 1.12, we have $\lfloor -x \rfloor = -\lceil x \rceil$.

   SOLUTION 2: Both $\lfloor -x \rfloor$ and $-\lceil x \rceil$ are integers satisfying $n \leq -x < n+1$. Because the floor of any real x is unique (by a theorem proved in class, part of the definition of "floor"), these integers must be equal.

5. Use induction to prove that for all $n \in \mathbb{N}$, $\binom{2n}{n} \leq 4^n$.

   Proof by vanilla induction. Base case $n = 1$: $\binom{2n}{n} = \binom{2}{1} = \frac{2!}{1!1!} = 2$, which is less than $4^n = 4$.

Now, let $n \in \mathbb{N}$ and suppose that $\binom{2n}{n} \leq 4^n$. We have $\binom{2(n+1)}{(n+1)} = \frac{(2n+2)!}{(n+1)!(n+1)!} = \frac{(2n+2)(2n+1)(2n)!}{(n+1)n!(n+1)n!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)}\binom{2n}{n} < \frac{(2n+2)(2n+2)}{(n+1)(n+1)}\binom{2n}{n} = \frac{2(n+1)2(n+1)}{(n+1)(n+1)}4^n = 4^{n+1}$. Hence $\binom{2(n+1)}{(n+1)} \leq 4^{n+1}$.

6. Solve the recurrence with initial conditions $a_0 = 1, a_1 = 4$ and relation $a_n = 3a_{n-1} - 2a_{n-2}$ $(n \geq 2)$.

Our characteristic polynomial is $r^2 - 3r + 2 = (r-2)(r-1)$, so the general solution is $a_n = A2^n + B1^n = A2^n + B$. We now apply the initial conditions to get $1 = a_0 = A2^0 + B = A + B$ and $4 = a_1 = A2^1 + B = 2A + B$. We now solve the system $\{A + B = 1, 2A + B = 4\}$ to get $A = 3, B = -2$. Hence the specific solution is $a_n = 3 \cdot 2^n - 2$.

7. Consider the nonstandard order $\prec$ on $\mathbb{Z}$ given by $0 \prec 1 \prec -1 \prec 2 \prec -2 \prec 3 \prec \cdots$. The smallest element is 0, the second smallest is 1. Find a formula for the n^{th} smallest element.

The n^{th} smallest element is $\begin{cases} n/2 & n \text{ is even} \\ (1-n)/2 & n \text{ is odd} \end{cases}$.

One can avoid cases, at the expense of a messier formula, e.g. $\left\lceil \frac{n(-1)^n}{2} \right\rceil$ or $(-1)^n \lfloor \frac{n}{2} \rfloor$.

8. Consider the sequence $a_n = 3n^2 + 100n + 1$. Prove that $a_n = \Theta(n^2)$.

We need to prove both $a_n = O(n^2)$ (harder) and $a_n = \Omega(n^2)$ (easier).

$a_n = O(n^2)$: Take $n_0 = 100, M = 5$. Let $n \geq n_0 = 100$. We have $|a_n| = |3n^2 + 100n + 1| = 3n^2 + 100n + 1 \leq 3n^2 + n^2 + n^2 = 5n^2 = M|b_n|$.

$a_n = \Omega(n^2)$: Take $n_0 = 1, M = 1$. Let $n \geq n_0 = 1$. We have $M|a_n| = 1|3n^2 + 100n + 1| = 3n^2 + 100n + 1 \geq n^2 = |b_n|$.

9. Prove that $\forall n \in \mathbb{N}_0$, the Fibonacci numbers F_n satisfy $F_n < 1.9^n$.

Proof by strong induction. We need two base cases: $F_0 = 0 < 1 = 1.9^0$, and $F_1 = 1 < 1.9^1$.

Now, let $n \in \mathbb{N}_0$, and assume that $F_n < 1.9^n$ and $F_{n+1} < 1.9^{n+1}$. We have $F_{n+2} = F_{n+1} + F_n < 1.9^{n+1} + 1.9^n = 1.9^n(1.9+1) = 1.9^n(2.9) < 1.9^n(3.61) = 1.9^n(1.9)^2 = 1.9^{n+2}$. Hence $F_{n+2} < 1.9^{n+2}$.

10. Find a recurrence relation for sequence T_n such that the Master Theorem would give $T_n = \Theta(\sqrt{n}\log n)$. Describe an algorithm that would satisfy your recurrence relation.

The form of the solution can only arise in the "middle c_n" case. Hence, $k = d = 1/2$. Hence $\log_b a = 1/2$ and $c_n = \Theta(\sqrt{n})$. One possibility is $a = 2, b = 4, c_n = \sqrt{n}$, which gives $T_n = 2T_{n/4} + \sqrt{n}$. An algorithm that satisfies this recurrence relation would divide the size n problem into four parts, call itself recursively on two of the parts, and have an overhead of $\sqrt{n}$ putting those two results together.